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ON A. ZYGMUND DIFFERENTIATION CONJECTURE

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ABSTRACT. Consider v a Lipschitz unit vector field on R^n and K its Lipschitz constant.

We show that the maps $S_s : S_s(X) = X + sv(X)$ are invertible for $0 \leq |s| < 1/K$ and define nonsingular point transformations. We use these properties to prove first the differentiation in L^p norm for $1 \leq p < \infty$. Then we show the existence of a universal set of values $s \in [-1/2K, 1/2K]$ of measure $1/K$ for which the Lipschitz unit vector fields $v \circ S_s^{-1}$ satisfy Zygmund's conjecture for all functions in $L^p(\mathbb{R}^n)$ and for each p , $1 \leq p < \infty$.

1. INTRODUCTION

Lebesgue differentiation theorem states that given a function $f \in L^1(\mathbb{R})$ the averages $\frac{1}{2t} \int_{-t}^t f(x+u)du$ converge a.e. to $f(x)$ when t tends to zero. The differentiation for functions F defined on \mathbb{R}^2 is more subtle. Actually it is a longstanding problem to find analogue of Lebesgue differentiation theorem for averages of the form

$$M_t(F)(x, y) = \frac{1}{2t} \int_{-t}^t F[(x, y) + \beta v(x, y)] d\beta$$

for a measurable function v . One would expect these averages to converge a.e. to $F(x, y)$. In other words one looks at the differentiation along the vector field v (or the direction v). (see for instance [6], [2]). One can see that because of the geometry of R^2 multiple directions are possible. In fact the example of the Nikodym set [2] shows that condition on

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v must be imposed if one expects the differentiation to hold. J. Bourgain [1] established the differentiation of the averages $M_t(F)$ for function $F \in L^2$ and v a real analytic vector field. N.H. Katz [4] has some partial result for Lipschitz vector fields. A longstanding conjecture attributed to A. Zygmund (see the paper by M. Lacey and X. Li, [5]) is the following.

Zygmund's conjecture

Let v be a Lipschitz unit vector field and let $F \in L^2(\mathbb{R}^2)$. Do the averages

$$M_t(F)(x, y) = \frac{1}{2t} \int_{-t}^t F[(x, y) + \beta v(x, y)] d\beta$$

converge a.e. to $F(x, y)$?

First we will observe that for s small enough (if K is the Lipschitz constant of v we will require $|s| < 1/2K$), the maps $S_s : S_s(x) = x + sv(x)$ are invertible. This observation will allow us to derive the norm convergence of the averages

$$M_t(F) = \frac{1}{2t} \int_{-t}^t F(x + \beta v(x)) d\beta$$

to the function F in all L^p spaces, $1 \leq p < \infty$. This norm convergence result was apparently an open problem (see [1].)

Then we will show that Zygmund's conjecture holds in all L^p spaces $1 \leq p < \infty$ for the unit vector fields $v \circ S_s^{-1}$ when $s \in \mathcal{T}$, a universal subset of $[-1/2K, 1/2K]$ with measure $1/K$. The method we use extends to \mathbb{R}^n .

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2. DIFFERENTIATION IN \mathbb{R}^2

The main steps are as follows. First we show that for s small enough the maps $S_s : S_s(x, y) = (x, y) + sv(x, y)$ are invertible and nonsingular in the sense that $\mu(A) = 0$ if and only $\mu(S_s(A)) = 0$. A more precise statement is given in Lemma 1 where we prove that the operators induced by these maps are uniformly bounded on $L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$. From this we derive the norm convergence of the averages $M_t(F)$ to F . Two consequences are derived from Lemma 1. First we obtain a "weak" version of our main result, Proposition 2, where we show that given a function $F \in L^1(\mathbb{R}^2)$ the differentiation occurs along the vector fields $v \circ S_s^{-1}$ as long as s belongs to a set of measure $1/K$ depending a priori on F . Then we use Hardy Littlewood maximal inequality on $L^1(\mathbb{R})$ to derive a first maximal inequality for the differentiation problem (Theorem 3). Our main result is proved by showing that the set where the differentiation occurs can in fact be taken independently of any $F \in L^1(\mathbb{R}^2)$. Finally we establish a "local" maximal inequality for the maximal operator associated with these averages.

2.1. Convergence in L^p norm.

Lemma 1. *Assume that v is a unit vector field (i.e. $\|v(x, y)\|_2 = 1$ for all $(x, y) \in \mathbb{R}^2$ and a Lipschitz map with constant K . Then for each t ; $|t| \leq T < \frac{1}{K}$ the map S_t from \mathbb{R}^2 to \mathbb{R}^2 such that $S_t(x, y) = (x, y) + tv(x, y)$ is one to one and onto. Furthermore if we denote by μ Lebesgue measure on \mathbb{R}^2 for all measurable sets $A \subset \mathbb{R}^2$, for all $|s| \leq T$, we have*

$$\frac{1}{2\pi(1 + |s|K)^2} \mu(S_s(A)) \leq \mu(A) \leq 2\pi \left(\frac{1}{1 - |s|K} \right)^2 \mu(S_s(A)).$$

Proof. First if $S_t(x_1, y_1) = S_t(x_2, y_2)$ then we have

$$\|(x_1, y_1) - (x_2, y_2)\| = \|t(v(x_1, y_1) - v(x_2, y_2))\| \leq KT\|(x_1, y_1) - (x_2, y_2)\|.$$

As $KT < 1$ this shows that S_t is one to one.

The equation $Z = (z_1, z_2) = (x, y) + tv(x, y) = X + tv(X)$ has a solution in $X = (x, y)$ that can be found by applying the fixed point theorem to the function $R_Z: R_Z(X) = Z + X - S_t(X)$.

To establish the second part of the lemma we can observe that it is enough to prove it for cubes A . For any two points $Z_1 = X_1 + sv(X_1)$ and $Z_2 = X_2 + sv(X_2)$ we have $\|Z_1 - Z_2\| \leq (1 + |s|K)\|X_1 - X_2\|$, and $\|X_1 - X_2\| \leq \frac{1}{1-|s|K}\|S_s(X_1) - S_s(X_2)\|$. Also for each measurable compact set $B \subset R^2$ we have $\mu(B) \leq \pi \text{diam}(B)^2$, where $\text{diam}(B)$ is the diameter of the bounded set B . As $\|S_s(X) - S_s(Y)\| \leq (1 + |s|K)\|X - Y\|$ we have $\text{diam}(S_s(A)) \leq (1 + |s|K)\text{diam}(A)$. Therefore if we denote by r the side length of the cube A we have

$$\mu(S_s(A)) \leq \pi \text{diam}(S_s(A))^2 \leq \pi(1+|s|K)^2 \text{diam}(A)^2 = \pi(1+|s|K)^2 2r^2 \leq 2\pi(1+|s|K)^2 \mu(A).$$

By approximation we conclude that for any measurable set A we have the same inequality.

From the inequality $\|X_1 - X_2\| \leq \frac{1}{1-|s|K}\|S_s(X_1) - S_s(X_2)\|$, we can conclude that

$$\|S_s^{-1}(Y_1) - S_s^{-1}(Y_2)\| \leq \frac{1}{1-|s|K}\|Y_1 - Y_2\|$$

for all $Y_1, Y_2 \in R^2$. The same path will lead us then to the inequality

$$\mu(S_s^{-1}(B)) \leq 2\pi\left(\frac{1}{1-|s|K}\right)^2 \mu(B)$$

for all measurable set $B \subset \mathbb{R}^2$. From this we can derive the second inequality in the lemma.

□

Using the notations of Lemma 1 we can obtain the convergence in L^p norm.

Proposition 1. *For $0 < |t| \leq T$ and for $1 \leq p \leq \infty$ the operators M_t defined pointwise by*

$$M_t(F)(x, y) = \frac{1}{2t} \int_{-t}^t F[(x, y) + sv(x, y)] ds$$

map L^p into L^p . Furthermore for each $1 \leq p < \infty$ we have

$$\lim_{t \rightarrow 0} \|M_t(F) - F\|_p = 0.$$

Proof. It follows immediately from Lemma 1. Indeed the case $p = \infty$ is obvious. For the other values of p , consider a nonnegative simple L^p integrable function $F = \sum_{n=1}^N \alpha_n \mathbf{1}_{A_n}$ with disjoint measurable sets A_n . We have

$$\begin{aligned} \|M_t(F)\|_p^p &= \int_{\mathbb{R}} \left| \frac{1}{2t} \int_{-t}^t \sum_{n=1}^N \alpha_n \mathbf{1}_{A_n}(S_s(x, y)) ds \right|^p d\mu \\ &\leq \int_{\mathbb{R}} \left(\frac{1}{2t} \int_{-t}^t \sum_{n=1}^N \alpha_n^p \mathbf{1}_{A_n}(S_s(x, y)) ds \right) ds d\mu = \frac{1}{2t} \int_{-t}^t \sum_{n=1}^N \alpha_n^p \mu(S_s^{-1}(A_n)) ds \\ &\leq \frac{1}{2t} \left(\int_{-t}^t 2\pi \left(\frac{1}{1 - |s|K} \right)^2 ds \right) \sum_{n=1}^N \alpha_n^p \mu(A_n) = \frac{2\pi}{1 - tK} \|F\|_p^p \end{aligned}$$

The boundedness of the operators M_t follows by approximation.

The second part of the proposition is a consequence of the simple fact that for the dense set of continuous functions with compact support we have the pointwise and norm convergence of the operators M_t .

□

2.2. A "weak" version of Zygmund's conjecture. The next proposition is a "weak" version of Zygmund's conjecture in the sense that for each function $F \in L^1(\mu)$ there exists a set of s of measure T such that

$$\lim_{t \rightarrow 0} \frac{1}{2t} \int_{-t}^t F[(x, y) + \beta v(S_s^{-1}x)] d\beta = F(x, y)$$

for almost every $(x, y) \in \mathbb{R}^2$. In other words the set of s and Lipschitz vector fields $v \circ S_s^{-1}$ for which the differentiation occurs may depend on F . The next proposition gives us also a path on how to approach Zygmund's conjecture, more precisely by considering the averages along the values of the function F at $(x, y) + \beta v(S_s^{-1}(x, y))$ and by exploiting the invertibility of the maps S_s .

Proposition 2. *Let v be a Lipschitz function from \mathbb{R}^2 to \mathbb{R}^2 with Lipschitz constant K such that $\|v(x, y)\|_2 = 1$ for almost all $(x, y) \in \mathbb{R}^2$. Then for all function $F \in L^1(\mathbb{R}^2)$ for almost every $s \in [-T/2, T/2]$, for almost every $(x, y) \in \mathbb{R}^2$ we have*

$$\lim_{t \rightarrow 0} \frac{1}{2t} \int_{-t}^t F[(x, y) + (s + \beta)v(x, y)] d\beta = F[(x, y) + sv(x, y)]$$

and

$$\lim_{t \rightarrow 0} \frac{1}{2t} \int_{-t}^t F[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta = F(x, y).$$

Proof. For t, s and β small enough we consider the averages

$$\frac{1}{2t} \int_{-t}^t F[(x, y) + (s + \beta)v(x, y)] d\beta$$

Because of the assumptions made on v by Lemma 1 for each $s; |s| \leq T < \frac{1}{K}$ for almost all $(x, y) \in \mathbb{R}^2$

$$G_{x,y}(s) = F[(x, y) + sv(x, y)]$$

is well defined and $G_{x,y} \in L^1([-T, T])$. By Lebesgue differentiation theorem, for almost every $s \in [-T/2, T/2]$ we have

$$\lim_t \frac{1}{2t} \int_{-t}^t F[(x, y) + (s + \beta)v(x, y)]d\beta = F[(x, y) + sv(x, y)].$$

Let us consider the complement E in $\mathbb{R}^2 \times [-T/2, T/2]$ of the set

$$\{(x, y, s) : \lim_t \frac{1}{2t} \int_{-t}^t F[(x, y) + (s + \beta)v(x, y)]d\beta = F[(x, y) + sv(x, y)]\}$$

By Fubini this set has measure zero. Again by Fubini for almost all s the set $E_s = \{(x, y) : (x, y, s) \in E\}$ also has measure zero. By lemma 1 the corresponding sets $S_s(E_s)$ will also have measure zero. This proves the second part of the proposition.

□

As indicated above the maximal inequality allowing to derive the conclusions of proposition 3 is given by the following result.

Theorem 3. *Let K be the Lipschitz constant for the unit vector field v . Then for each T , $0 < T < 1/K$, for all $\lambda > 0$*

$$\begin{aligned} & \frac{1}{T} \int_{-T/2}^{T/2} \mu \left\{ (x, y) \in \mathbb{R}^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]|d\beta > \lambda \right\} dm(s) \\ & \leq \frac{4\pi^2(1 + TK)^2}{(1 - TK)^2} \frac{1}{\lambda} \int_{\mathbb{R}^2} |F(x, y)|d\mu. \end{aligned}$$

where m denotes Lebesgue measure on $[-T/2, T/2]$.

Proof. For a.e. (x, y) the function $G(x, y) : G_{x,y}(s) = \mathbf{1}_{[-T, T]}(s) \cdot F[(x, y) + sv(x, y)]$ belongs to L^1 . By Hardy-Littlewood maximal inequality applied to this function we have;

$$\begin{aligned} & m \left\{ s \in [-T/2, T/2] : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\} \\ & \leq \frac{1}{\lambda} \int_{-T}^T |F[(x, y) + \beta v(x, y)]| d\beta. \end{aligned}$$

We can integrate both sides of this inequality with respect to Lebesgue measure μ on \mathbb{R}^2 and apply Fubini theorem.

We obtain by using Lemma 1,

$$\begin{aligned} & \mu \times m \left\{ (x, y, s) \in R^2 \times [-T/2, T/2] : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\} \\ & \leq \frac{1}{\lambda} \int_{-T}^T \int_{\mathbb{R}^2} |F[(x, y) + \beta v(x, y)]| d\mu d\beta \\ & \leq \frac{2\pi}{(1 - TK)^2 \lambda} \int_{-T}^T \int_{\mathbb{R}^2} |F(x, y)| d\mu d\beta \\ & = \frac{2\pi T}{(1 - TK)^2 \lambda} \int_{\mathbb{R}^2} |F(x, y)| d\mu = \frac{CT}{\lambda} \int_{\mathbb{R}^2} |F(x, y)| d\mu \end{aligned}$$

Dividing all expressions above by T and rewriting

$$\mu \times m \left\{ (x, y, s) \in R^2 \times [-T/2, T/2] : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\}$$

as

$$\int_{-T/2}^{T/2} \mu \left\{ (x, y) \in R^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\} dm(s),$$

we derive the following inequality:

$$\begin{aligned} & \frac{1}{T} \int_{-T/2}^{T/2} \mu \left\{ (x, y) \in R^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\} dm(s) \\ & \leq \frac{C}{\lambda} \int_{\mathbb{R}^2} |F(x, y)| d\mu. \end{aligned}$$

Using Lemma 1 we can observe that

$$\begin{aligned} & \mu \left\{ (x, y) \in \mathbb{R}^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + (s + \beta)v(x, y)]| d\beta > \lambda \right\} \\ & \geq \frac{1}{2\pi(1 + TK)^2} \mu \left\{ (x, y) \in \mathbb{R}^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\}. \end{aligned}$$

Therefore, for all $\lambda > 0$, we have the inequality

$$\begin{aligned} & \frac{1}{T} \int_{-T/2}^{T/2} \mu \left\{ (x, y) \in \mathbb{R}^2 : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} dm(s) \\ & \leq \frac{4\pi^2(1 + TK)^2}{(1 - TK)^2} \frac{1}{\lambda} \int_{\mathbb{R}^2} |F(x, y)| d\mu. \end{aligned}$$

□

2.3. A universal set of unit Lipschitz vector fields satisfying Zygmund's conjecture in all L^p spaces. As indicated in the introduction we want to strengthen Proposition 2 by showing that a universal set of vector fields $v \circ S_s^{-1}$ satisfy Zygmund's conjecture. More precisely we want to prove the following result.

Theorem 4. *Let v be a unit Lipschitz vector field with Lipschitz constant $K = 1/T$. Then there exists a set $\mathcal{T} \subset [-T/2, T/2]$ of measure T such that for each $s \in \mathcal{T}$ the unit Lipschitz vector field $v \circ S_s^{-1}$ satisfies Zygmund's conjecture in all L^p spaces for $1 \leq p < \infty$. More precisely for all $F \in L^p(\mathbb{R}^2)$ the averages*

$$\frac{1}{2t} \int_{-t}^t F[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta$$

converge a.e to $F(x, y)$.

To prove this theorem we introduce some notation. We denote by $\mathcal{D}_N = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| \leq N\}$ and by \mathcal{E} a countable set of continuous functions with compact support dense

in the unit closed ball of $L^1(\mathbb{R}^2)$. We will use the notation $M_t^s(F)(x, y)$ for the averages

$$\frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta$$

Theorem 4 is a consequence of the following result.

Theorem 5. *Under the assumptions of Theorem 4, we have for each p , $1 \leq p < \infty$ and for a.e. $s \in [-T/2, T/2]$*

$$\lim_{n \rightarrow \infty} \sup_{\|F\|_p \leq 1} \mu\{(x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n\} = 0.$$

Our proof of these theorems will require several lemmas. We only give the proof for the case $p = 1$. The case $p > 1$ can be obtained similarly without difficulty as the differentiation is a local property. Given any function $F \in L^1(\mu)$ there exists a subsequence $G_j = F_{N_j}$ such that $\lim_j G_j(x, y) = F(x, y)$ except on a set of measure zero \mathcal{N} . The next lemma is a consequence of Lemma 1. It shows that for almost every (x, y) we can keep this convergence along the line segments $[(x, y) - \beta w(x, y), (x, y) + \beta w(x, y)]$ where β is in absolute value smaller than the reciprocal of the Lipschitz constant of the unit vector field w .

Lemma 2. *Let $F \in L^1(\mu)$ and G_j a sequence of continuous function with compact support converging a.e. to F . Let w be a unit vector field with Lipschitz constant \mathbf{K} . Then for almost all $x \in \mathbb{R}^2$ for all $t \in [-T/2, T/2]$ we have*

$$\begin{aligned} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta w(x, y)]| d\beta &= \frac{1}{2t} \int_{-t}^t \liminf_j |G_j|[(x, y) + \beta w(x, y)] d\beta \\ &= \sup_j \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta \end{aligned}$$

Proof. Let us consider the null set \mathcal{N} off which the sequence G_j converges to F . We can assume that this set is measurable. Hence by Fubini we have

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{-T}^T \mathbf{1}_{\mathcal{N}}[(x, y) + \beta w(x, y)] d\beta d\mu &= \int_{-T}^T \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{N}}[(x, y) + \beta w(x, y)] d\beta d\mu \\ &\leq \left(\frac{1}{1-TK}\right)^2 2T\mu(\mathcal{N}) = 0. \end{aligned}$$

Therefore there exists a set A of zero measure such that for $(x, y) \in A^c$ we have

$$\int_{-T}^T \mathbf{1}_{\mathcal{N}}[(x, y) + \beta w(x, y)] d\beta = 0. \text{ Hence for}$$

all $t \in [-T/2, T/2]$ we also have $\int_{-t}^t \mathbf{1}_{\mathcal{N}}[(x, y) + \beta w(x, y)] d\beta = 0$. Writing the function

$|F|$ as $\mathbf{1}_{\mathcal{N}}|F| + \mathbf{1}_{\mathcal{N}^c}|F|$ we have then for $(x, y) \in A^c$

$$\begin{aligned} \int_{-t}^t |F[(x, y) + \beta w(x, y)]| d\beta &= \int_{-t}^t \mathbf{1}_{\mathcal{N}^c}[(x, y) + \beta w(x, y)] |F[(x, y) + \beta w(x, y)]| d\beta \\ &= \int_{-t}^t \mathbf{1}_{\mathcal{N}^c}[(x, y) + \beta w(x, y)] \liminf_j |G_j|[(x, y) + \beta w(x, y)] d\beta \\ &= \lim_j \frac{1}{2t} \int_{-t}^t \mathbf{1}_{\mathcal{N}^c}[(x, y) + \beta w(x, y)] \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta \end{aligned}$$

by the monotone convergence theorem

$$= \lim_j \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta$$

□

Next we want to check that the preceding lemma applies to all Lipschitz unit vector fields $v \circ S_s^{-1}$.

Lemma 3. *Let v be a Lipschitz unit vector field with Lipschitz constant K . Consider $0 < T < 1/K$. Then for all s such that $0 < |s| < T/2$ the unit vector fields $v \circ S_s^{-1}$ are Lipschitz vector fields with Lipschitz constant $2K$.*

Proof. As v is a Lipschitz vector field with Lipschitz constant K we have for all X, Y in \mathbb{R}^2

$$\|v(S_s^{-1})(X) - v(S_s^{-1})(Y)\| \leq K\|S_s^{-1}(X) - S_s^{-1}(Y)\|.$$

We denote $Z_1 = S_s^{-1}(X)$ and $Z_2 = S_s^{-1}(Y)$. Then $X = Z_1 + sv(Z_1)$ and $Y = Z_2 + sv(Z_2)$.

Therefore $\|Z_1 - Z_2\| \leq \|X - Y\| + |s|K\|Z_2 - Z_1\|$ and we obtain

$$\|Z_1 - Z_2\| \leq \frac{1}{(1 - |s|K)}\|X - Y\|. \text{ We conclude then that}$$

$$\|v(S_s^{-1})(X) - v(S_s^{-1})(Y)\| \leq K \frac{1}{(1 - |s|K)}\|X - Y\|.$$

Noticing that for $0 < |s| < T/2$ we have $\frac{1}{(1 - |s|K)} \leq 2$ and this concludes the proof of this lemma. \square

Thus we can apply Lemma 2 with the constant $\mathbf{K} = 2K$.

Lemma 4. *For each $\lambda > 0$ and each $s \in [-T/2, T/2]$ we have*

$$(1) \quad \begin{aligned} & \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} \\ &= \sup_{\Phi_j \in \mathcal{E}} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |\Phi_j[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} \end{aligned}$$

Proof. Let us fix $\varepsilon > 0$. We can find a function $F \in L^1$ with $\|F\|_1 \leq 1$ such that

$$(2) \quad \begin{aligned} & \sup_{\|G\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |G[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} \\ & \leq \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} + \varepsilon \end{aligned}$$

For the function F we can find a subsequence $G_j = F_{n_j}$ of continuous functions in \mathcal{E} which converges a.e. to F . Applying Lemma 2 with $w = v \circ S_s^{-1}$ off a null set \mathcal{N}_s we have

$$\begin{aligned}
& \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta w(x, y)]| d\beta = \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \liminf_j |G_j|[(x, y) + \beta w(x, y)] d\beta \\
&= \sup_{0 < t < T/2} \sup_j \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta \\
&= \sup_j \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta \\
&= \lim_j \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta w(x, y)] d\beta
\end{aligned}$$

(Noticing that the sup is the limit because we have an increasing sequence)

Hence we have

$$\begin{aligned}
& \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |F[(x, y) + \beta v(S_s^{-1}(x, y))]| d\beta > \lambda \right\} \\
&= \mu \left\{ (x, y) \in \mathcal{D}_N : \lim_j \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta > \lambda \right\} \\
&= \lim_j \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta > \lambda \right\} \\
&\leq \limsup_j \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t \inf_{k \geq j} |G_k|[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta > \lambda \right\} \\
&\leq \sup_{\Phi \in \mathcal{E}} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} \frac{1}{2t} \int_{-t}^t |\Phi|[(x, y) + \beta v(S_s^{-1}(x, y))] d\beta > \lambda \right\}
\end{aligned}$$

This last inequality combined with (2) proves Lemma 4. \square

Lemma 5. *For F continuous with compact support and each $\lambda > 0$ the map*

$$s \in [-T/2, T/2] \mapsto \mu \left\{ (x, y) \in \mathcal{D}_N ; \sup_{0 < t < T/2} M_t^s(F)(x, y) > \lambda \right\}$$

is continuous.

Proof. Again we denote by X the vector $(x, y) \in \mathbb{R}^2$. For all $|\beta| \leq T/2$ and for all $|s_1|, |s_2| \leq T/2$ we have

$$\|(X + \beta v(S_{s_1}^{-1}X)) - (X + \beta v(S_{s_2}^{-1}X))\| \leq \frac{T}{2}K\|S_{s_1}(X) - S_{s_2}(X)\|.$$

For $Z_1 = S_{s_1}(X)$ and $Z_2 = S_{s_2}(X)$ we have

$$Z_1 + s_1 v(Z_1) = X = Z_2 + s_2 v(Z_2).$$

Therefore we have

$$Z_1 - Z_2 = s_2 v(Z_2) - s_1 v(Z_1) = (s_2 - s_1)v(Z_2) + s_1(v(Z_2) - v(Z_1)).$$

As a consequence we obtain

$$\|Z_1 - Z_2\| \leq |s_2 - s_1| + \frac{T}{2}K\|Z_1 - Z_2\|$$

and this gives us the uniform estimate

$$\|(X + \beta v(S_{s_1}^{-1}X)) - (X + \beta v(S_{s_2}^{-1}X))\| \leq \frac{KT}{2}(1 - \frac{TK}{2})|s_1 - s_2| = C|s_1 - s_2|.$$

Now we can conclude by using the uniform continuity of the function F . For $|s_1 - s_2| < \frac{\delta(\varepsilon)}{C}$

then for all $X \in R^2$ we have

$$\left| \sup_{0 < t < T/2} M_t^{s_1} F(X) - \sup_{0 < t < T/2} M_t^{s_2} F(X) \right| < \varepsilon.$$

□

The following lemma is well known and can be found in [3]. We just state it to make the paper hopefully easier to read.

Lemma 6. *Let \mathcal{C} be any collection of open intervals B in \mathbb{R} and let U be the union of all these open intervals. If $c < m(U)$, then there exist disjoint $B_1, \dots, B_k \in \mathcal{C}$ such that*

$$\sum_{j=1}^k m(B_j) > \frac{1}{3}c.$$

Now we can proceed with the proof of Theorem 5. For simplicity we will denote by $M_*^s(F)(X)$ the maximal function

$$\sup_{0 < t < T/2} M_t^s(F)(x, y)$$

Proof of Theorem 5

We will argue by contradiction. Because of Lemma 4 the functions

$$H_n : s \in [-T/2, T/2] \rightarrow H_n(s) = \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\}$$

being equal for each s to

$$\sup_{F_i \in \mathcal{E}} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F_i)(x, y) > n \right\}$$

are measurable and decreasing with n . If the conclusion of Theorem 5 was false then we could find a measurable set $A \subset (-T/2, T/2)$ with positive measure and a positive number δ such that for each $s \in A$ and for each $n \in \mathbb{N}$ we would have

$$(3) \quad H_n(s) > \delta$$

We can observe that the set A can be written as

$$A = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > n \right\} > \delta \right\}.$$

For each n the set

$$\bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > n \right\} > \delta \right\},$$

being open by Lemma 5, it is a countable union of disjoint open intervals. Therefore the collection (with n) of all these intervals is countable. Because of the decreasing nature of the sets

$$\bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > n \right\} > \delta \right\}$$

with n , the intervals obtained at stage $k+1$ are included in those corresponding to stage k .

Our goal is to find a more appropriate countable covering of A . First we can pick an integer N_1 large enough and an increasing sequence of integers $(N_k)_{k \geq 1}$ such that the following conditions are satisfied.

(1)

$$A \subset V_{N_1} = \bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > N_1 \right\} > \delta \right\}.$$

(2)

$$m \left\{ \bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > N_1 \right\} > \delta \right\} \right\} \leq 2m(A)$$

(3) $\sum_{k=1}^{\infty} \frac{k}{N_k} \leq \left(\frac{\delta}{3}\right)^2 m(A) \gamma$, where the constant γ will be specified later in order to establish a contradiction.

To start the selection process we pick any $s_1 \in A$. Then there exists an open interval $I_{1,N_1} \subset V_{N_1}$ that contains s_1 . Then we pick $s_2 \in A \cap I_{1,N_1}^c$ and select I_{1,N_2} containing s_2 . By induction we can obtain a countable collection of open intervals $\mathcal{J}_1 = \bigcup_{k=1}^{\infty} I_{1,N_k} \subset V_{N_1}$. If

this collection does not cover A then we continue the selection process by picking $s' \in A \cap \mathcal{J}_1^c$

and an open interval

$$I_{2,N_2} \subset V_{N_2} = \bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > N_2 \right\} > \delta \right\} \subset V_{N_1}$$

that contains s' . The difference between the collections \mathcal{J}_1 and \mathcal{J}_2 is that the first is built with the sequence $(N_k)_{k \geq 1}$ while the second starting with N_2 is built with the sequence $(N_{k+1})_{k \geq 1}$. Because, as we noticed above, we started with at most countably many open intervals and that at each step we picked a different open interval, the selection process has to stop after countably many iterations. So we obtain after induction at most a countable number of collections \mathcal{J}_r , $r \in \mathbb{N}$, that will cover A and will all be contained in V_{N_1} .

We denote the union of these collections of sets by $\mathcal{R} = \bigcup_{r=1}^{\infty} \mathcal{J}_r$. We can observe that with this selection process we have at most one interval associated with N_1 , two with N_2 and generally at most k with N_k . Now we can use Lemma 6 to extract of this collection of open intervals, disjoint open intervals G_1, G_2, \dots, G_R such that

$$(**) \sum_{h=1}^R m(G_h) > \frac{1}{3}m(A).$$

As all these intervals are disjoint subsets of V_{N_1} we also have

$$(***) \sum_{h=1}^R m(G_h) \leq 2m(A).$$

Now we can reach a contradiction. We combine what we obtained so far to make our choice of γ . We have

$$\frac{\delta}{3}m(A) \leq \int_{-T/2}^{T/2} \sum_{h=1}^R \mathbf{1}_{G_h}(s) \mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m_h})(X) > \Gamma_h \right\} ds$$

for some integers m_h and Γ_h ,

$$\leq \sum_{h=1}^R (m(G_h))^{1/2} \left(\int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m_h})(X) > \Gamma_h \right\} \right)^2 ds \right)^{1/2}$$

by Cauchy Schwartz's inequality,

$$\begin{aligned} &\leq \sum_{h=1}^R (m(G_h))^{1/2} (\mu(\mathcal{D}_N))^{1/2} \left(\int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m_h})(X) > \Gamma_h \right\} \right) ds \right)^{1/2} \\ &\leq \left(\sum_{h=1}^R m(G_h) \right)^{1/2} (\mu(\mathcal{D}_N))^{1/2} \left(\sum_{h=1}^R \int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m_h})(X) > \Gamma_h \right\} \right) ds \right)^{1/2} \end{aligned}$$

by Cauchy Schwartz's inequality,

$$\leq (2m(A))^{1/2} \mu(\mathcal{D}_N)^{1/2} T^{1/2} \left(\sum_{h=1}^R \frac{4\pi^2(1+TK)^2}{(1-TK)^2} \frac{1}{M_h} \right)^{1/2}$$

by using (***), Theorem 3 and the fact that $\|F_{m_h}\|_1 \leq 1$

Therefore we have

$$\begin{aligned} \frac{\delta}{3}(m(A))^{1/2} &\leq T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \left(\sum_{h=1}^R \frac{1}{\Gamma_h} \right)^{1/2} \\ &\leq T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \left(\sum_{h=1}^{\infty} \frac{h}{N_h} \right)^{1/2} \end{aligned}$$

because we had for each k at most k intervals corresponding to N_k

$$< T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \frac{\delta}{3} (m(A))^{1/2} \gamma^{1/2}$$

by using (3).

To establish a contradiction it is enough now to pick

$$\gamma < \frac{1}{T\mu(\mathcal{D}_N)\frac{8\pi^2(1+TK)^2}{(1-TK)^2}}$$

choice that we could have made independently of the selection process. This ends the proof of Theorem 5.

Because of Lemma 1 Theorem 5 can be reformulated in the following way

Corollary 1. *Let v be a unit Lipschitz vector field with Lipschitz constant $K = 1/T$. Then there exists a set $\mathcal{T} \subset [-T/2, T/2]$ of measure T such that for each $s \in \mathcal{T}$, for all $F \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$, the averages*

$$\frac{1}{2t} \int_{-t}^t F[(x, y) + (\beta + s)v(x, y)]d\beta$$

converge a.e. to $F[(x, y) + sv(x, y)]$.

Proof of Theorem 4

Theorem 5 provides us with a set \mathcal{T} of measure T such that for each $s \in \mathcal{T}$ we have

$$\lim_n \sup_{\|F\|_1 \leq 1} \mu \left\{ X \in \mathcal{D}_N : M_*^s(F)(X) > n \right\} = 0.$$

We can conclude, by using similar arguments as those displayed in [3], that for each $s \in \mathcal{T}$ the set of functions in $L^1(\mathbb{R}^2)$ for which the pointwise convergence holds on \mathcal{D}_N is closed in L^1 . As we obviously have the dense set of continuous functions for which the differentiation holds we have proved Theorem 4 for values of $X \in \mathcal{D}_N$. The general case follows by letting N tend to infinity.

2.4. A maximal inequality for the unit vector field $v \circ S_s^{-1}$. First we want to refine the proof of Theorem 5 in order to evaluate the rate of convergence to zero (with n) of the maximal function $\sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\}$.

Theorem 6. *For each $0 < \alpha < 1/2$, for a.e. s in a set of measure T in $[-T/2, T/2]$ we have*

$$\lim_n n^\alpha \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\} = 0.$$

Proof. As in Theorem 5 we argue by contradiction. Instead of the functions H_n we use this time the functions

$$O_n : s \in [-T/2, T/2] \rightarrow O_n(s) = n^\alpha \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\}$$

which for each s are equal to

$$n^\alpha \sup_{F_i \in \mathcal{E}} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\}$$

(by Lemma 4.) The set that replaces A is

$$B = \left\{ s \in (-T/2, T/2) : \limsup_n O_n(s) > \delta \right\}$$

By Lemma 5, for each positive integer L the set

$$W_L = \bigcap_{n=1}^L \bigcup_{j \geq n} \bigcup_{i=1}^\infty \left\{ s \in (-T/2, T/2) : j^\alpha \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > j \right\} > \delta \right\}$$

as a finite intersection of open sets is a countable union of disjoint open intervals. By taking the collection (with L) of all these open intervals we obtain a countable number of such intervals. As before the intervals obtained at stage $L+1$ are subsets of those corresponding

to stage L . Having a countable number of intervals we proceed with an increasing sequence of integers N_k such that

$$(4) \quad \sum_{k=1}^{\infty} \frac{k}{N_k^{1-2\alpha}} \leq \left(\frac{\delta}{3}\right)^{2m(B)\gamma'}$$

We can start the selection process with the additional conditions

$$(5) \quad B \subset W_{N_1} = \bigcap_{n=1}^{N_1} \bigcup_{j \geq n} \bigcup_{i=1}^{\infty} \left\{ s \in (-T/2, T/2) : \mu \left\{ (x, y) \in \mathcal{D}_N : M_*^s(F_i)(x, y) > j \right\} > \delta \right\}$$

$$(6) \quad m(W_{N_1}) \leq 2m(B)$$

As before we select by induction a covering $\mathcal{R}' = \bigcup_{r'=1}^{\infty} \mathcal{I}'_r$ of B by open intervals, subsets of those composing W_{N_1} . Furthermore in this entire collection \mathcal{R}' of open intervals we have at most one associated with N_1 , two with N_2 and more generally at most k with N_k . Next we use Lemma 6 to extract of this collection, disjoint open intervals, G'_1, G'_2, \dots, G'_R such that

$$+++ \frac{1}{3}m(B) < \sum_{h=1}^R m(G'_h) \leq 2m(B), \text{ as these intervals are disjoint subsets of } W_{N_1}.$$

To establish the contradiction we will choose later γ' appropriately. We have

$$\frac{\delta}{3}m(B) \leq \int_{-T/2}^{T/2} \sum_{h=1}^R \mathbf{1}_{G'_h}(s) (\Gamma'_h)^{\alpha} \mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m'_h})(X) > \Gamma'_h \right\} ds$$

for some integers m'_h and Γ'_h .

We can use Cauchy Schwarz's inequality to dominate this last term.

$$\begin{aligned}
&\leq \sum_{h=1}^R (m(G'_h))^{1/2} (\Gamma'_h)^\alpha \left(\int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m_h})(X) > \Gamma'_h \right\} \right)^2 ds \right)^{1/2} \\
&\leq \sum_{h=1}^R (m(G_h))^{1/2} (\Gamma'_h)^\alpha (\mu(\mathcal{D}_N))^{1/2} \left(\int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m'_h})(X) > \Gamma'_h \right\} \right) ds \right)^{1/2} \\
&\leq \left(\sum_{h=1}^R m(G'_h) \right)^{1/2} (\mu(\mathcal{D}_N))^{1/2} \left(\sum_{h=1}^R (\Gamma'_h)^{2\alpha} \int_{-T/2}^{T/2} \left(\mu \left\{ X \in \mathcal{D}_N; M_*^s(F_{m'_h})(X) > \Gamma'_h \right\} \right) ds \right)^{1/2}
\end{aligned}$$

by Cauchy Schwartz's inequality,

$$\leq (2m(B))^{1/2} \mu(\mathcal{D}_N)^{1/2} T^{1/2} \left(\sum_{h=1}^R \frac{4\pi^2(1+TK)^2}{(1-TK)^2} \frac{(\Gamma'_h)^{2\alpha}}{\Gamma'_h} \right)^{1/2}$$

by using (+++) and Theorem 3

$$= (2m(B))^{1/2} \mu(\mathcal{D}_N)^{1/2} T^{1/2} \left(\sum_{h=1}^R \frac{4\pi^2(1+TK)^2}{(1-TK)^2} \frac{1}{(\Gamma'_h)^{1-2\alpha}} \right)^{1/2}.$$

Hence we have

$$\begin{aligned}
\frac{\delta}{3} (m(B))^{1/2} &\leq T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \left(\sum_{h=1}^R \frac{1}{\Gamma_h^{1-2\alpha}} \right)^{1/2} \\
&\leq T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \left(\sum_{h=1}^{\infty} \frac{h}{N_h^{1-2\alpha}} \right)^{1/2}
\end{aligned}$$

because we had for each k at most k intervals corresponding to N_k

$$< T^{1/2} \mu(\mathcal{D}_N)^{1/2} \left(\frac{8\pi^2(1+TK)^2}{(1-TK)^2} \right)^{1/2} \frac{\delta}{3} (m(B))^{1/2} \gamma'^{1/2}$$

by using (4).

The contradiction is obtained for

$$\gamma' < \frac{1}{T \mu(\mathcal{D}_N) \frac{8\pi^2(1+TK)^2}{(1-TK)^2}}.$$

□

The following result can be derived from Theorem 5.

Theorem 7. *For each $0 < \alpha < 1/2$ there exists a function C_α almost everywhere finite on $[-T/2, T/2]$ such that for all $\lambda > 1$, for all $F \in L^1$ such that $\lambda > \|F\|_1$ we have*

$$(7) \quad \mu \left\{ X \in \mathcal{D}_N : M_*^s(F)(X) > \lambda \right\} \leq 2^\alpha C_\alpha(s) \left(\frac{\|F\|_1}{\lambda} \right)^\alpha$$

Proof. For a fixed α we denote by $C_{n,\alpha}(s)$ the a.e. finite function

$$n^\alpha \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > n \right\}.$$

By Theorem 6 we have $\lim_n C_{n,\alpha}(s) = 0$ for a.e. $s \in [-T/2, T/2]$. Hence the function $C_\alpha : C_\alpha(s) = \sup_n C_{n,\alpha}(s)$ is a.e. finite on $[-T/2, T/2]$. Furthermore for each $\lambda > 1$ we have

$$\begin{aligned} & \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > \lambda \right\} \\ & \leq \sup_{\|F\|_1 \leq 1} \mu \left\{ (x, y) \in \mathcal{D}_N : \sup_{0 < t < T/2} M_t^s(F)(x, y) > [\lambda] \right\} \\ & \leq \frac{C_\alpha(s)}{[\lambda]^\alpha} \end{aligned}$$

Therefore by using the inequality $\lambda < 2[\lambda]$ we have

$$\sup_{\|F\|_1 \leq 1} \mu \left\{ X \in \mathcal{D}_N : M_*^s(F)(X) > \lambda \right\} \leq \frac{2^\alpha C_\alpha(s)}{\lambda^\alpha}$$

Now by taking the functions $F/\|F\|_1$ and making the change $\lambda\|F\|_1 = t$ we can derive (7).

□

3. DIFFERENTIATION IN R^n

The results obtained in the previous section can be extended without difficulty to \mathbb{R}^n . In fact the only part where we use the fact that we were in \mathbb{R}^2 is when we proved Lemma 1. This appears in the constant of this Lemma. We only state the lemma that would replace Lemma 1 in \mathbb{R}^n . We consider a Lipschitz unit vector field v on R^n with constant K and simply denote by $M_t(F)(X)$ the averages

$$\frac{1}{2t} \int_{-t}^t F[X + \beta v(X)] d\beta$$

where $X = (x_1, x_2, \dots, x_n)$. We use the same notation for $S_s(X) = X + \beta v(X)$ a map from R^n to R^n . We denote by μ_n Lebesgue measure on R^n .

Lemma 7. *For all $|s| \leq T$ where $T < 1/K$ the maps S_s are one to one and onto. Furthermore there exist constants c_n and C_n depending only on n , K and T such that for all measurable sets A in R^n we have*

$$c_n \mu_n(S_s(A)) \leq \mu_n(A) \leq C_n \mu_n(S_s(A))$$

Proof. The invertible character of the maps S_s for small s can be established in the same way. The inequalities

$$c_n \mu_n(S_s(A)) \leq \mu_n(A) \leq C_n \mu_n(S_s(A))$$

follow from the inequalities $\|Z_1 - Z_2\| \leq (1 + |s|K)\|X_1 - X_2\|$, and $\|X_1 - X_2\| \leq \frac{1}{1 - |s|K} \|S_s(X_1) - S_s(X_2)\|$ where $Z_1 = S_s(X_1)$ and $Z_2 = S_s(X_2)$. \square

The maximal inequality that replaces Theorem 3 is the following

Theorem 8. *Let K be the Lipschitz constant for the unit vector field v . Then for each T ,*

$0 < T < 1/K$, there exists a constant \mathcal{C}_n such that for all $\lambda > 0$

$$\begin{aligned} & \frac{1}{T} \int_{-T/2}^{T/2} \mu_n \left\{ X \in \mathbb{R}^n : \sup_{0 < t \leq T/2} \frac{1}{2t} \int_{-t}^t |F[X + \beta v(S_s^{-1}(X))]| d\beta > \lambda \right\} dm(s) \\ & \leq \frac{\mathcal{C}_n}{\lambda} \int_{\mathbb{R}^n} |F(X)| d\mu_n. \end{aligned}$$

where m denotes Lebesgue measure on $[-T/2, T/2]$.

The proof is identical to the one given for Theorem 3 so we skip it. From this maximal inequality the reasoning is identical. The only difference is the constant \mathcal{C}_n that depends on T , K and n . From that point on by using the same path one can extend Theorem 4 and Theorem 5 to the case of \mathbb{R}^n .

REFERENCES

- [1] **J. Bourgain**: “A remark on the maximal function associated to an analytic vector field”, *Proc. Of the Special Year in Modern Analysis*, London Math. Soc. Lect. note Series, vol 137, 111-132, 1989.
- [2] **M. De Guzman**: “Real variable methods in Fourier Analysis”, *Math. Study*, 46, North Holland, 1981.
- [3] **G. Folland**: “Real Analysis”, *J. Wiley and Sons*, 1984.
- [4] **N.H. Katz**: “A partial result on Lipschitz differentiation”, *Harmonic Analysis at Mount Holyoke*, Contemporary Math. , vol 320, 217-224, 2003.
- [5] **M. Lacey and X. Li**: “On the Hilbert Transform and $C^{1+\varepsilon}$ Families of Lines”, *Annals of Math.*, to appear.
- [6] **E.M. Stein**: “Real variable methods, orthogonality, and oscillatory integrals”, *Princeton Math. Series*, 43, 1993.
- [7] **N. Wiener** : “The ergodic theorem”, *Duke Math. J.*, 5, 1-18, 1939.